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Unsteady slot suction from a high-Reynolds-number cross-flow

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Abstract. The problem of unsteady suction from a high-Reynolds-number cross-flow into a slot is considered in the case where the suction is driven by a time-dependent slot pressure. The model uses linearised asymptotics based on a small parameter that defines the suction strength. An integro differential equation is derived for the mass flow into the slot and this is solved for various time-dependent slot pressures of practical interest. Closed-form expressions are also found for the shape of the shear layer dividing the external flow from the fluid in the slot. For a step function change in the slot pressure, a non-monotonic decay to the steady solution is observed, and for an oscillatory slot pressure there is a phase lag between the slot pressure and the mass flow. For rapidly changing slot pressures it is shown that slot *injection* can occur, even when the slot pressure remains below the free-stream pressure.

Keywords: suction, injection, film cooling, Hilbert transforms, thin aerofoil theory.

1. Introduction

The problem of slot suction from a cross-flow into a slot is known to be relevant to the study of film cooling of turbine blades. In film cooling, cool air is emitted from the surface of a turbine blade, and forms an insulating layer along the surface of the blade, thus permitting higher turbine entry temperatures to be used. The cooling air is sucked from a channel within the blade, and it is this slot suction that is considered here. Of particular interest is the mass flow into the slot. Slot suction is also relevant to the so-called 'rim seal' problem (see Chew *et al.* [1]) where, because of the passage of turbine blades above it, the gap between the turbine rotor and stator periodically injects and sucks fluid to and from the free stream. In this case, the main concern is to prevent large unsteady suction forces that would allow hot free stream gas to damage delicate mechanisms at the bottom of the gap.

The analysis given here is a generalisation of the limiting weak suction model of Dewynne *et al.* [2], who considered steady slot suction, driven by maintaining a constant static pressure at the bottom of the slot below the value of the static pressure of the cross-flow. This model in turn used the results of Michell [3], who considered flow through an aperture in a wall dividing a uniform flow from a stagnant region which was maintained at a lower pressure than the uniform flow. Michell deduced a relationship between the angle through which the flow through the aperture is turned and the static pressure drop across the wall far downstream, finding that an infinite pressure drop was required to turn the flow through a right-angle. This suggests that for a finite pressure drop, the streamline that separates at the slot leading edge cannot reattach to the upstream slot wall. The results of Michell [3] and Dewynne *et al.* [2] further assert that the streamline that divides the flow ingested into the slot from the free

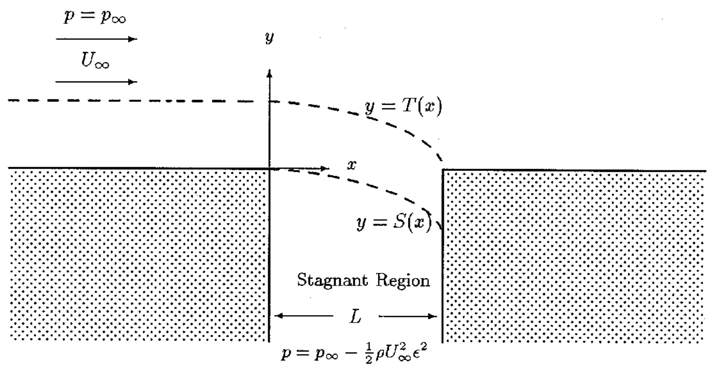

Figure 1. A schematic representation of steady slot suction.

stream must attach either to the downstream wall of the slot or to the wall downstream of the slot trailing edge. For the weak suction considered here, the latter of these cases is prohibited.

For strong suction, *i.e.* suction where the pressure difference is order one, Dewynne *et al.* found analytic expressions for the velocity by use of a hodograph transformation. For weak suction they used a linearised theory, and thus found expressions that agreed with those for strong suction in the asymptotic limit. Good agreement with experiment was found for both strong and weak slot suction (see Morland [4]).

For steady weak suction, a small parameter ε, a measure of the suction strength, is defined by asserting that the static pressure in the slot (of width L) is $p_\infty - \frac{1}{2}\rho U_\infty^2\varepsilon^2$, where p_∞, ρ and U_∞ are respectively the pressure, density and speed of the undisturbed cross-flow. A schematic representation of the flow is shown in Figure 1.

The shear layer that divides the external flow from the stagnant flow in the slot is denoted by $y = S(x)$, whilst $y = T(x)$ identifies the boundary between flow that is eventually ingested into the slot and free stream flow. For the steady case, both S and T are streamlines of the flow. In Figure 1, $S(x)$ is shown attaching to the downstream slot wall. In reality, it is evident that such attachment cannot happen and S must tend to $-\infty$ as $x \to L$. However it was shown in the steady case that $S(x)$ remains finite as $x \to L$. This apparent anomaly is caused by the fact that the linearised model is not valid within a distance of $O(L\varepsilon)$ from the downstream wall of the slot. However an analysis of the flow in this thin region is not necessary for the purposes of this problem, so S is considered to be finite for $x \in [0, L]$.

It is well known from thin aerofoil theory (see, for example Van Dyke [5, pp. 45–76]) that an order ε^2 pressure perturbation produces a disturbance of thickness order ε^2. Since the velocity of the fluid that is sucked into the slot from the free stream is of order U_∞, the mass flow into the slot is therefore of order $LU_\infty\rho\varepsilon^2$. For the analysis below to be valid, ε must be small so that linear asymptotics may be applied, and the shear layer $y = S(x, t)$ must be of negligible width so that viscous effects may be ignored. Since the shear layer is of thickness $O(L\,\mathrm{Re}^{-1/2})$, and the height of the dividing streamline is $O(L\varepsilon^2)$, the conditions for this model to apply are

$$\mathrm{Re}^{-1/4} \ll \varepsilon \ll 1.$$

2. Unsteady slot suction

To deal with the case of unsteady slot suction, some changes must be made to Figure 1. S and T are now functions of x and t, and need no longer be streamlines of the flow. The free-stream speed U_∞ is still assumed to be constant, but we introduce time dependence by assuming that the slot pressure (though still constant across the slot) is given by $p_\infty - \frac{1}{2}\rho U_\infty^2 \varepsilon^2 f(t)$, where $f(t)$ is an order one function of time. The previous steady case analysed in Dewynne *et al.* [2] may therefore be recovered by setting $f(t) = 1$.

The external flow is irrotational, and to lowest order is simply uniform flow in the x-direction. The suction into the slot may be modelled as a distribution of sinks of strength $\varepsilon^2\gamma(x,t)$, say, situated along the x-axis between $x = 0$ and $x = L$. The velocity potential Φ of the flow is therefore given by

$$\Phi = U_\infty x + \frac{U_\infty \varepsilon^2}{2\pi} \int_0^L \gamma(\xi, t) \log\left(\frac{(x-\xi)^2 + y^2}{L^2}\right) \mathrm{d}\xi. \tag{1}$$

We may determine the function γ in terms of S by observing that the dividing shear layer, $y = S(x,t)$, must satisfy the kinematic condition

$$\frac{\mathrm{D}}{\mathrm{D}t}(y - S(x,t)) = 0, \quad \text{on } y = S(x,t).$$

Substitution of the velocities u and v in the above equation gives (since $u = U_\infty$ to lowest order)

$$\Phi_y = U_\infty S_x + S_t + O(\varepsilon^2), \quad \text{on } y = S(x,t). \tag{2}$$

The orders of magnitude of the terms in this equation depend on the time scale of changes in the slot pressure function. The most interesting case occurs when changes in the slot pressure take place over a time scale of order L/U_∞. Changes over a longer time scale lead to a quasi-steady problem, in which t behaves merely as a parameter. If a shorter time scale is considered, the fluid outside the slot is not able to move a significant distance in this time, so that the flow in the external region, and therefore the shear layer, does not change over this time scale. Therefore the effects of changes over a shorter time scale are confined to the slot.

The relevant non-dimensionalisation for pressure fluctuations occurring on a time scale L/U_∞ is

$$x = Lx^*,$$

$$y = \begin{cases} Ly^* & \text{(outer flow)}, \\ L\varepsilon^2 y^* & (S(x,t) \leqslant y \leqslant T(x,t)), \end{cases}$$

$$S(x,t) = L\varepsilon^2 S^*(x^*, t^*),$$

$$t = LU_\infty^{-1} t^*,$$

$$\Phi(x,y,t) = LU_\infty x^* + LU_\infty \varepsilon^2 \Phi^*(x^*, y^*, t^*),$$

$$p = p_\infty + \tfrac{1}{2}\rho U_\infty^2 \varepsilon^2 p^*.$$

It is also convenient to define a variable σ via $\sigma_x = S$, with $\sigma = 0$ at $x = 0$. The mass flow into the slot, measured across the top of the slot will be denoted by $\rho M_0(t)$. As we discussed earlier, the mass flow is $O(LU_\infty \rho \varepsilon^2)$, so the relevant non-dimensionalisation of these two variables is given by

$$\begin{aligned} \sigma(x,t) &= L^2\varepsilon^2\sigma^*(x^*,t^*), \\ M_0(t) &= LU_\infty\varepsilon^2 M_0^*(t^*). \end{aligned}$$

A relationship between σ and M_0 may be deduced from (2), since the mass flow at the top of the slot may be found by integration of the vertical velocity across the top of the slot. As a positive mass flow into the slot corresponds to a negative vertical velocity at the entrance to the slot, the mass flow is

$$LU_\infty\varepsilon^2 M_0^*(t^*) = -\int_0^L \Phi_y \,\mathrm{d}x = -LU_\infty\varepsilon^2(\sigma^*_{x^*}(1,t^*) + \sigma^*_{t^*}(1,t^*)). \tag{3}$$

Working in the non-dimensional variables defined above, we may evaluate the derivatives of Φ from (1). In particular, in the limit as y tends to zero we have

$$\begin{aligned} \lim_{y\to 0^+} \Phi_y &= \lim_{y\to 0^+} \frac{U_\infty\varepsilon^2}{\pi}\int_0^1 \gamma(\xi,t^*)\frac{y^*}{(x^*-\xi)^2+y^{*2}}\,\mathrm{d}\xi \\ &= U_\infty\varepsilon^2\int_0^1 \gamma(\xi,t^*)\delta(x^*-\xi)\,\mathrm{d}\xi \\ &= U_\infty\varepsilon^2\gamma(x^*,t^*). \end{aligned}$$

In the external flow, where y is scaled with L, to lowest order the shear layer is given by $y = 0$. Thence from (2)

$$\gamma = S^*_{x^*} + S^*_{t^*} = \sigma^*_{x^*x^*} + \sigma^*_{x^*t^*}, \tag{4}$$

to leading order.

Equations (1) and (4) give the velocity potential in terms of σ^* only. Hence the pressure may be evaluated just above the shear layer from Bernoulli's equation. To do this, it is necessary to evaluate Φ_x and Φ_t from (1) in the limit as y tends to zero. Thus,

$$\begin{aligned} \lim_{y\to 0^+} \Phi_x &= \lim_{y\to 0^+} U_\infty + \frac{U_\infty\varepsilon^2}{\pi}\int_0^1 (\sigma^*_{\xi\xi} + \sigma^*_{\xi t^*})\frac{x^*-\xi}{(x^*-\xi)^2+y^{*2}}\,\mathrm{d}\xi \\ &= U_\infty - \frac{U_\infty\varepsilon^2}{\pi} ⨍_0^1 \frac{\sigma^*_{\xi\xi}+\sigma^*_{\xi t^*}}{\xi - x^*}\,\mathrm{d}\xi, \end{aligned}$$

where $⨍$ denotes a Cauchy Principal Value integral. Similarly, we may find Φ_t, using (3), and the boundary conditions $\sigma = \sigma_x = 0$, to give

$$\lim_{y\to 0^+} \Phi_t = \lim_{y\to 0^+} \frac{U_\infty^2\varepsilon^2}{2\pi}\int_0^1 (\sigma^*_{\xi\xi} + \sigma^*_{\xi t^*})_{t^*}\log((x^*-\xi)^2+y^{*2})\,\mathrm{d}\xi,$$

$$= \frac{U_\infty^2 \varepsilon^2}{\pi}[(\sigma^*_{\xi t^*} + \sigma_{t^* t^*}) \log|x^* - \xi|]_0^1$$

$$-\frac{U_\infty^2 \varepsilon^2}{\pi} \lim_{y \to 0^+} \int_0^1 \frac{(\sigma^*_{\xi t^*} + \sigma_{t^* t^*})(\xi - x^*)}{(x^* - \xi)^2 + y^{*2}} \,\mathrm{d}\xi,$$

$$= \frac{U_\infty^2 \varepsilon^2}{\pi} \left(M_0^{*\prime}(t^*) \log|1 - x^*| - ⨍_0^1 \frac{\sigma^*_{\xi t^*} + \sigma_{t^* t^*}}{\xi - x^*} \,\mathrm{d}\xi \right). \tag{5}$$

The $\log|1-x^*|$ term suggests that unless M_0' is zero Φ_t changes by an infinite amount between the slot and infinity. This shows that for a time-dependent mass flow it is not possible to impose the conditions $p = p_\infty$ and $u = U_\infty$ at infinity, since, in an unsteady system with the assumptions made here, this would imply that the total pressure would tend to infinity. Therefore, the boundary condition we impose at infinity will be that the total pressure is constant, which implies that the pressure differs from its original value, p_∞, by order $\varepsilon^2 \log x^*$ at large distances.

Here, we denote the total pressure at infinity as $p_\infty + \frac{1}{2}\rho U_\infty^2$, so Bernoulli's equation is

$$\frac{p}{\rho} + \tfrac{1}{2}\rho(\Phi_x^2 + \Phi_y^2) + \rho\Phi_t = p_\infty + \tfrac{1}{2}\rho U_\infty^2.$$

The zero-order terms in the above equation cancel, as the equation must be considered to order ε^2. The term Φ_y^2 is $O(\varepsilon^4)$, according to (2), and so may be ignored. This gives the non-dimensionalised pressure just above the shear layer, which, as the pressure is continuous across the shear layer and is constant in the stagnant region below the shear layer, is just given by $p^* = -f(t)$. Hence,

$$-f(t^*) = \frac{2}{\pi} ⨍_0^1 \frac{\sigma^*_{\xi\xi} + 2\sigma^*_{\xi t^*} + \sigma^*_{t^* t^*}}{\xi - x^*} \,\mathrm{d}\xi + \frac{2}{\pi} M_0^{*\prime}(t^*) \log(1 - x^*). \tag{6}$$

Since, by using (3), we may express M_0^* in terms of σ^*, Equation (6) may be thought of as an equation in σ^* alone. It is convenient to express all three terms of (6) as singular integrals. The equations may then be written as

$$-\frac{1}{\pi} ⨍_0^1 \left(f(t^*) \sqrt{\frac{\xi}{1 - \xi}} \right) \frac{\mathrm{d}\xi}{\xi - x^*}$$

$$= \frac{2}{\pi} ⨍_0^1 \left(\sigma^*_{\xi\xi} + 2\sigma^*_{\xi t^*} + \sigma^*_{t^* t^*} + \frac{M_0^{*\prime}(t^*)}{\pi} b(\xi) \right) \frac{\mathrm{d}\xi}{\xi - x^*},$$

where $b(x)$ is the function whose finite-range Hilbert transform is $\log|1 - x|$, namely (see Lattimer [6, pp. 152–153] for details)

$$b(x^*) = 2 \arcsin \sqrt{x^*} - 2\sqrt{\frac{x^*}{1 - x^*}} \log 2.$$

As the same integral transform occurs on both sides it follows that

$$-f(t)\sqrt{\frac{x^*}{1-x^*}} = 2\left(\sigma^*_{x^*x^*} + 2\sigma^*_{x^*t^*} + \sigma^*_{t^*t^*} + \frac{M_0^{*\prime}(t^*)}{\pi}b(x^*)\right) + h(x^*), \tag{7}$$

where $h(x^*)$ must satisfy

$$\int_0^1 h(\xi)\frac{\mathrm{d}\xi}{\xi - x^*} = 0.$$

Since it is known (see, for example, Muskhelishvili [7, pp. 249–252]) that the only solutions for h are proportional to $\{x^*(1-x^*)\}^{-1/2}$, whilst σ and its first two derivatives are zero at $x^* = 0$, it follows that h must be identically zero.

For a given M_0, Equation (7) is a parabolic second-order partial differential equation which may be solved for σ^* in terms of M_0 by changing variables from (x^*, t^*) to $(\xi, \eta) = (x^* + t^*, x^* - t^*)$ and integrating twice with respect to ξ. This gives

$$\begin{aligned}\sigma^* \;=\; & a_1(x^* - t^*) + (x^* + t^*)a_2(x^* - t^*) \\ & - \int_0^{x^*}\int_0^{x_1} \tfrac{1}{2} f(x_2 - x^* + t^*)\sqrt{\frac{x_2}{1-x_2}}\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ & - \frac{1}{\pi}\int_0^{x^*}\int_0^{x_1} M_0^{*\prime}(x_2 - x^* + t^*)b(x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1,\end{aligned} \tag{8}$$

where a_1 and a_2 are arbitrary functions which may be determined from the boundary conditions. In this case, since the boundary conditions are $\sigma = \sigma_{x^*} = 0$ at $x^* = 0$, it follows that a_1 and a_2 are identically equal to zero.

Finally, on substitution of (8) in (3), we obtain an integro differential equation for $M_0(t^*)$ in the form

$$M_0^*(t^*) = \int_0^1 \tfrac{1}{2} f(x_1 - 1 + t^*)\sqrt{\frac{x_1}{1-x_1}}\,\mathrm{d}x_1 + \frac{1}{\pi}\int_0^1 M_0^{*\prime}(x_1 - 1 + t^*)b(x_1)\,\mathrm{d}x_1. \tag{9}$$

This equation will henceforth be referred to as the mass-flow equation. (For convenience, the asterisks will henceforth be omitted.) Once (9) has been solved to determine M_0, the function σ may be recovered from (8). This allows S and therefore T and the pressure to be determined. Note that (9) has been derived under the assumption that $f(t)$ is defined for $-\infty < t < \infty$. Cases where $f(t) = 0$ for $t < 0$ may be considered by introducing the appropriate Heaviside functions in (9).

3. Solution of the mass-flow equation

We will now show that the integro differential equation (9) may be solved in closed form for a wide range of potentially useful slot pressure functions $f(t)$. In general, the easiest way to proceed is to use integral transform methods, but after examining cases where this is necessary, we briefly consider some special cases where particularly simple methods of solution are possible.

3.1. Solution by means of integral transforms

To determine a solution of (9) for general $f(t)$, transform methods may be employed. We wish particularly to examine cases where steady suction occurs until time $t = 0$, after which time-dependent flows are produced by varying slot pressure functions. In such cases, (which include physically interesting and industrially relevant examples such as $H(t), tH(t)$, $H(t)\sin\nu t$ where $H(t)$ denotes the Heaviside function) it is most convenient to use a Laplace transform, and it will henceforth be assumed that the slot pressure function possesses such a transform. It is also possible to consider slot pressure functions defined for $t < 0$ (for example, cases such as $\sin\nu t$ and $\delta(t)$) by using a Fourier transform, but we will not pursue this further here.

To solve using a Laplace transform, we observe that the two integrals in (9) may be written as convolution integrals. If $f(t)$ is zero for $t < 0$ and we define the function $\alpha(\xi)$ by

$$\alpha(1-\xi) = \begin{cases} \sqrt{\frac{\xi}{1-\xi}} & 0 < \xi < 1, \\ 0 & \xi \leqslant 0, \xi \geqslant 1, \end{cases}$$

then the first integral in (9) may be written

$$\frac{1}{2}\int_{-\infty}^{\infty} f(\xi - 1 + t)\alpha(1-\xi)\,\mathrm{d}\xi.$$

Making the substitution $u = 1 - \xi$, we note that for $u > t$ the argument of f is negative, so that f is zero. Furthermore, if u is negative, then $\alpha(u)$ is zero, so that the integrand can only be nonzero for u between 0 and t. Hence the above integral is simply

$$\frac{1}{2}\int_0^t f(t-u)\alpha(u)\,\mathrm{d}u,$$

the Laplace convolution of f and α. Similarly, the second integral in (9) may be expressed as the convolution of M_0' and $b(1-x)$.

From the results

$$\int_0^1 \sqrt{\frac{1-x}{x}}\,\mathrm{e}^{-px}\,\mathrm{d}x = \tfrac{1}{2}\pi\,\mathrm{e}^{-p/2}(I_0(\tfrac{1}{2}p) + I_1(\tfrac{1}{2}p)),$$

$$\int_0^1 b(1-x)\,\mathrm{e}^{-px}\,\mathrm{d}x = \pi\,\mathrm{e}^{-p/2}\left(\frac{\mathrm{e}^{p/2} - I_0(\frac{1}{2}p)}{p} - \log 2(I_0(\tfrac{1}{2}p) + I_1(\tfrac{1}{2}p))\right),$$

where I_0 and I_1 denote the usual modified Bessel functions, we may use (9) to obtain an expression for the Laplace transform $\overline{M}_0(p)$ of $M_0(t)$,

$$\overline{M}_0(p) = \tfrac{1}{4}\pi\overline{f}(p)\frac{I_0(\frac{1}{2}p) + I_1(\frac{1}{2}p)}{I_0(\frac{1}{2}p) + p\log 2(I_0(\frac{1}{2}p) + I_1(\frac{1}{2}p))}. \tag{10}$$

This is valid when $f(t)$ and $M_0(t)$ are zero for $t < 0$. (The linearity of the mass-flow equation means that cases where steady suction occurs for $t < 0$ may be dealt with by subtracting off the steady suction solution.) A similar expression may be obtained for the Fourier transform case.

In order to find M_0 it is necessary to determine the poles of $\overline{M}_0$ and evaluate the residues. It is clear from (10) that poles of $\overline{M}_0$ consist of the poles of $\overline{f}(p)$, and, since I_0 and I_1 are analytic everywhere, the zeroes of the denominator of the right-hand side. Writing

$$g(p) = I_0(\tfrac{1}{2}p) + p\log 2(I_0(\tfrac{1}{2}p) + I_1(\tfrac{1}{2}p)),$$

we note that, since the function $g(p)$ satisfies $g(\overline{p}) = \overline{g}(p)$, the poles occur in conjugate pairs. We write the poles as $-\lambda_i \pm i\mu_i$ where the λ_i are all real and the μ_i are strictly positive real numbers. The roots must be evaluated numerically, and, using the Newton-Raphson method, 320 pairs of simple roots were found. The corresponding values of λ_i and μ_i for the first 6 roots are shown in Table 1.

Table 1. The first 6 roots of $g(p)$.

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
λ_i	2·530	3·774	4·277	4·609	4·857	5·056
μ_i	1·983	8·976	15·409	21·763	28·088	34·400

It is relatively easy to show (for details see Lattimer [6, pp. 88–92]) that there are an infinite number of poles, all of which lie in the left-hand half of the complex plane. Moreover, it is important to establish whether $g(p)$ has any repeated roots, as this affects the form of the solution for $M_0(t)$. If there were a repeated root then $g'(p)$ would have to equal zero at a root of $g(p)$. In other words the equations

$$g(p) = (1 + p\log 2)I_0(\tfrac{1}{2}p) + (p\log 2)I_1(\tfrac{1}{2}p) = 0,$$

$$g'(p) = (\log 2 + \tfrac{1}{2}p\log 2)I_0(\tfrac{1}{2}p) + (\tfrac{1}{2} + \tfrac{1}{2}p\log 2)I_1(\tfrac{1}{2}p) = 0,$$

must have a simultaneous solution. If this were true, then

$$\frac{1 + p\log 2}{p\log 2} = \frac{2\log 2 + p\log 2}{1 + p\log 2}.$$

The solution to the above equation is real, and so cannot be a root of $g(p) = 0$, since $g(p)$ has no real roots. Hence, the functions $g(p)$ and $g'(p)$ have no common roots and so $g(p)$ has no repeated roots.

The sum of the residues of $e^{pt}\overline{M}_0(p)$ may now be determined, giving

$$M_0(t) = \tfrac{1}{4}\pi\sum_{n=1}^{\infty} e^{-\lambda_n t}(\alpha_n\cos\mu_n t + \beta_n\sin\mu_n t) + F(t), \tag{11}$$

where the α_n and β_n are real numbers given by

$$\alpha_n - i\beta_n = 4\overline{f}(p_n)\frac{I_0(\tfrac{1}{2}p_n) + I_1(\tfrac{1}{2}p_n)}{g'(p_n)}, \tag{12}$$

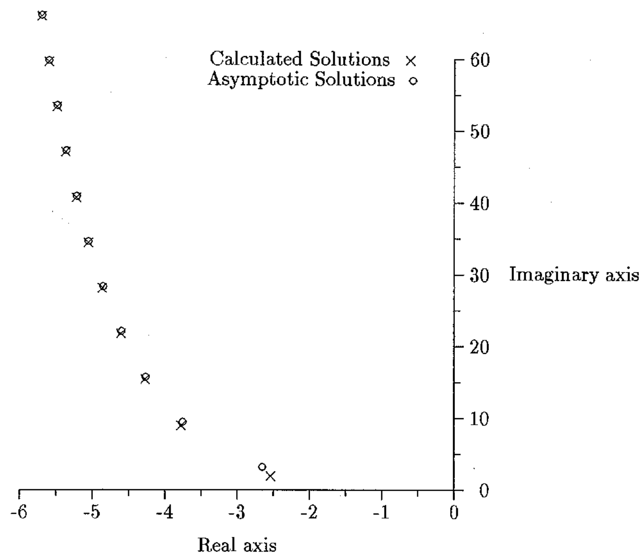

Figure 2. The first eleven poles of $g(p)$ in the upper-half plane.

and $F(t)$ encompasses the contributions from the residues of $e^{pt}\overline{M}_0(p)$ arising from poles of $\overline{f}(p)$.

Although formally (9) is now solved, further properties of the zeros of $g(p)$ and the poles of specific $\overline{f}(p)$ need to be determined if the convergence of (11) is to be analysed. In particular, the behaviour of λ_n and μ_n for large n is of interest. From the asymptotic formulae

$$J_0(z) \sim \sqrt{\frac{2}{\pi z}}\left(\sin(z+\tfrac{1}{4}\pi) - \frac{1}{8z}\cos(z+\tfrac{1}{4}\pi) + o(z^{-2}\sin z)\right),$$

$$J_1(z) \sim \sqrt{\frac{2}{\pi z}}\left(\cos(z+\tfrac{1}{4}\pi) - \frac{3}{8z}\sin(z+\tfrac{1}{4}\pi) + o(z^{-2}\sin z)\right),$$

(valid for $|\arg z| < \pi$), and the observation that $I_0(iz) = J_0(z)$ and $I_1(iz) = iJ_1(z)$ (see, for example Abramowitz and Stegun [8, pp. 355–434] for details), it may be shown that the asymptotic form of the roots of $g(p)$ is given by

$$\mu_n = (2n-1)\pi + O(n^{-1}),$$

$$\lambda_n = \log\left(\frac{2(2n-1)\pi\log 2}{1-\log 2}\right) + O(n^{-1}).$$

Values of this asymptotic profile are plotted against the calculated values of $-\lambda_i + i\mu_i$ in Figure 2, and these show that the calculated values tend quickly to the asymptotic behaviour.

This asymptotic behaviour is crucial in the determination of the convergence of the sum of the residues at p_n, *i.e.*

$$\sum_{n=1}^{\infty} e^{-\lambda_n t}(\alpha_n\cos\mu_n t + \beta_n\sin\mu_n t).$$

Clearly, it is necessary that this series be convergent for the above results to be meaningful. Moreover, if the position of the shear layer is to be calculated, then it is necessary that derivatives of the series also converge for a given $f(t)$. On substitution of the appropriate asymptotic expansion for the modified Bessel functions in (12), we obtain the asymptotic expression

$$\alpha_n - i\beta_n = 4\overline{f}(p)\frac{i}{ip_n \log 2 + \mathrm{e}^{-p_n}\log 2 - \frac{1}{2}\mathrm{e}^{-p_n}}.$$

Substitution of $p_n = -\lambda_n + i\mu_n$ in the above equation gives an expression for α_n and β_n which is of leading order $\overline{f}(p_n)/n$, since for large values of n the p_n resemble odd multiples of $n\pi$. Hence, the sum of the α_n and β_n is convergent for any function $f(t)$, provided $\overline{f}(p)$ tends to zero for large p at least as quickly as p^{-k} for some positive k, which will be true for any function whose Laplace transform exists. The functions $f(t) = H(t)$, $f(t) = tH(t)$, $f(t) = H(t)\sin \nu t$ all satisfy this condition and so the mass flow is finite. However, it is not always possible to differentiate (11) term by term, since with each differentiation each term is multiplied by a factor μ_n. For example, when $f(t) = H(t)$, the α_n are $O(n^{-2})$ and so the terms $\mu_n\alpha_n$ are $O(n^{-1})$ and the series does not converge at $t = 0$. However, the term $\mathrm{e}^{-\lambda_n t}$ is $O(n^{-t})$ (as $\lambda = O(\log \mu) = O(\log n)$) and so, for $t > 0$, the series converges. Hence, the expression obtained for the exponentially decaying term in the mass flow is differentiable for all positive t, but not at $t = 0$.

It is worth pointing out that there are other ways of writing (11). It follows immediately from (10) that M_0 may be written as a convolution of $f(t)$ and another function, $\zeta(t)$ say, in the form

$$M_0(t) = \int_0^t f(u)\zeta(t-u)\,\mathrm{d}u. \tag{13}$$

It therefore follows that $\zeta(t)$ is the solution for M_0 when $f(t)$ is a delta function, *i.e.* $\overline{f}$ is identically equal to 1. This solution, from (10), is just the sum of the residues of $\mathrm{e}^{pt}\overline{M}_0$, and so is given by

$$\zeta(t) = \tfrac{1}{4}\pi \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n t}(\alpha_n \cos \mu_n t + \beta_n \sin \mu_n t), \tag{14}$$

where the α_n and β_n are real numbers given by (12) with $\overline{f}$ identically equal to 1.

In many cases (13) provides a simpler way of obtaining the solution for $M_0(t)$.

3.2. Separation-of-variables solution of the mass-flow equation

The method of separation of variables may be used to solve (9) in a particularly simple way when the function $f(t)$ satisfies the condition that $f(x - t + 1)$ may be written as a linear combination of a finite number of separable functions, so that

$$f(x - 1 + t) = \sum_{i=1}^{n} a_i(x)b_i(t),$$

and additionally all of the functions $b_i(t)$ satisfy the condition

$$b_i'(t) = \sum_{j=1}^{n} c_{ij} b_j(t),$$

for some constants c_{ij}. In this case the solution is $M_0 = \sum A_i b_i(t)$, where the A_i may be found by direct substitution as solutions of a set of linear equations. Examples of possible functions $f(t)$ for which this method is applicable include polynomials, trigonometric and exponential functions. Sometimes, the solution takes a particularly simple form. For example the solution for $f(t) = t$ is

$$M_0(t) = \tfrac{1}{4}\pi t + \tfrac{1}{16}\pi - \tfrac{1}{4}\pi \log 2. \tag{15}$$

Although this case may be viewed as being unrealistic since for $t < 0$ blowing, rather than suction, is implied, it seems likely that, when t is large and positive, the dependence on the values of $f(t)$ for $t < 0$ will be small. We may confirm this by comparing (15) with the full solution when $f(t) = tH(t)$ which is given later by (21). If required, we could now calculate $S(x, t)$ using (8) and the pressure for $x < 0$ and $x > 1$ could be determined using (6), thus completely determining the flow.

A more physically realistic case that may be solved by means of separation of variables is given by the slot pressure function $f(t) = 1 + \sin \nu t$, where ν is a positive constant. This may be thought of as an idealised 'rim seal' slot pressure function where a steady slot suction is modified in a periodic fashion due to the passage of turbine blades in the vicinity of the slot. The mass flow in this case is given by

$$M_0(t) = A_\nu \cos \nu t + B_\nu \sin \nu t + \tfrac{1}{4}\pi,$$

where

$$A_\nu = \tfrac{1}{4}\pi \frac{J_0(\frac{1}{2}\nu)J_1(\frac{1}{2}\nu) - \nu \log 2 (J_0^2(\frac{1}{2}\nu) + J_1^2(\frac{1}{2}\nu))}{J_0^2(\frac{1}{2}\nu) - 2\nu \log 2 J_0(\frac{1}{2}\nu)J_1(\frac{1}{2}\nu) + \nu^2(\log 2)^2(J_0^2(\frac{1}{2}\nu) + J_1^2(\frac{1}{2}\nu))}, \tag{16}$$

$$B_\nu = \tfrac{1}{4}\pi \frac{J_0^2(\frac{1}{2}\nu)}{J_0^2(\frac{1}{2}\nu) - 2\nu \log 2 J_0(\frac{1}{2}\nu)J_1(\frac{1}{2}\nu) + \nu^2(\log 2)^2(J_0^2(\frac{1}{2}\nu) + J_1^2(\frac{1}{2}\nu))}. \tag{17}$$

This expression may also be obtained by the use of Fourier transforms.

We note that it may be shown that for all real positive ν $A_\nu < 0$ and $B_\nu \geqslant 0$. Hence, the mass flow and the pressure difference are never in phase. The phase lag takes a value between 0 and $\frac{1}{2}\pi$ for all ν. A phase lag of $\frac{1}{2}\pi$ only occurs when 2ν is a root of the Bessel function J_0 and a phase lag of zero only occurs for $\nu = 0$. In the limit as $\nu \to \infty$ both A and B tend to zero, with $\nu A_\nu = O(1) = \nu^2 B_\nu$ in the limit; for large ν the phase lag therefore tends to $\frac{1}{2}\pi$.

4. Examples of industrially relevant slot pressure functions

4.1. Net mass-flow maximisation

In some applications of unsteady slot suction the mass flow has to be maximised for a given 'suction cost'. One advantage of the convolution form of (13) is that it gives a direct expression for the mass flow in terms of the slot pressure. By integrating this equation we can show that the total mass flow up to time t is given by

$$\int_0^t M_0(s)\,\mathrm{d}s = \int_0^t f(u)Z(t-u)\,\mathrm{d}u, \tag{18}$$

where $Z(t)$ is defined by $Z(0) = 0$ and $Z'(t) = \zeta(t)$. This implies (using (14)) that Z tends to a constant as $t \to \infty$, and indicates that this decay is exponential. Furthermore, since ζ is the mass-transfer function for $f(t) = \delta(t)$, evidently (by differentiation of (13)) Z is the solution when $f(t) = H(t)$. Therefore, in the limit as $t \to \infty$, Z must tend to the value of the mass transfer in the steady state, which is $\frac{1}{4}\pi$.

In order to evaluate the function $\zeta(t)$, it is necessary to address the case when the slot pressure function is given by $f(t) = \delta(t)$. The solution in this instance is plotted in Figure 3. This shows that an instantaneous change in the slot pressure results in a mass flux that is not itself a delta function, but decays exponentially to zero. For large t, the first term of the sum in (14) dominates, and the decay constant is thus λ_1. It is of interest to note that the oscillatory nature of the solution means that the mass flow becomes negative at some time which will henceforth be denoted by t_c (t_c has been calculated to be approximately 1·642). This suggests that, even for a non-negative suction strength, fluid may be injected into the free stream. Clearly if this occurs a thin region of *injected* fluid must be present along the positive x-axis, rendering the model invalid. The fact that the solution for $\delta(t)$ becomes negative also implies that the derivative of the solution for $H(t)$ is negative at the corresponding value of t, a fact that may be confirmed by inspection of Figure 4.

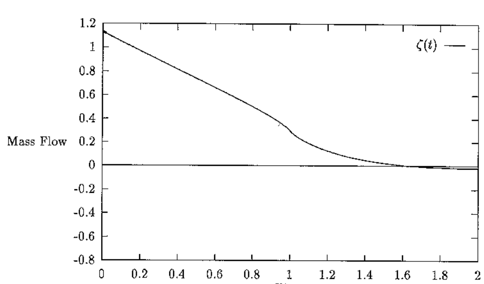

Figure 3. The mass flow $\zeta(t)$ when $f(t)$ is a Dirac delta function.

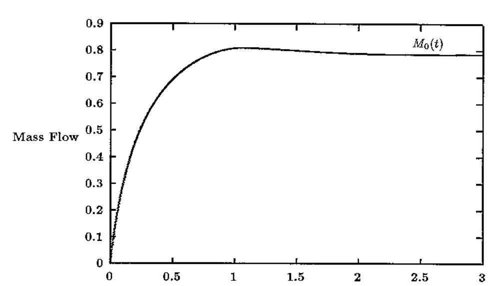

Figure 4. Mass flow when $f(t) = H(t)$.

From (18) it now follows that for any slot pressure function $f(t)$ that is bounded and integrable over the range $(0, \infty)$, the total mass flow always takes the same value, namely

$$\int_0^\infty M_0(u)\,\mathrm{d}u = \tfrac{1}{4}\pi \int_0^\infty f(u)\,\mathrm{d}u. \tag{19}$$

To show this, we re-write (18) as

$$\int_0^t M_0(u)\,\mathrm{d}u = \int_0^t f(u)(Z(t-u) - \tfrac{1}{4}\pi)\,\mathrm{d}u + \tfrac{1}{4}\pi \int_0^t f(u)\,\mathrm{d}u.$$

We will now consider the first integral separately over the regions $(0, K)$ and (K, t) for some $K > 0$. Since $Z - \frac{1}{4}\pi$ decays to zero exponentially for large t, for K sufficiently large there is an L such that $|Z(t) - \frac{1}{4}\pi| < L\,\mathrm{e}^{-\lambda_1 t}$ for $t > K$. It then follows that the integral of $f(u)(Z(t-u) - \frac{1}{4}\pi)$ over the region $(0, K)$ is bounded by $L \sup\{f(u)\}(\mathrm{e}^{-\lambda_1 K} - \mathrm{e}^{-\lambda_1 t})$, and so tends to zero if f is bounded. Over the region (K, t) the integrand is bounded above by

$$\sup\{Z - \tfrac{1}{4}\pi\} \int_K^t f(u)\,\mathrm{d}u$$

which tends to zero as K and t tend to infinity if f is integrable. Hence, the first integral is zero, and (19) holds. Therefore, for a given value of the integral of the slot pressure function, the total mass flow is always the same.

If total mass flow after a *finite* time is the main matter of concern, then for a given ‘cost’ in the integral of the pressure function, the total mass flow need not always be the same. This leads to a variational problem, where the quantity

$$\tfrac{1}{4}\pi \int_0^t f(u) Z(t-u)\,\mathrm{d}u$$

must be maximised subject to the constraint that $\int_0^t f(u)\,\mathrm{d}u = 1$, say. Clearly, the maximum total mass flow after a finite time will be achieved when f is a Dirac delta function centred around the value of u that maximises the value of $Z(t-u)$ over $(0, t)$. If $t < t_c$, (the time at which ζ first changes sign) then this maximum will occur at $u = 0$, so the maximal mass flow occurs for $f(t) = \delta(t)$. On intuitive grounds this is the expected result, as by concentrating the suction strength at the beginning of the time interval, there is more time to ingest the fluid. However, for $t \geqslant t_c$ the maximum of $Z(t-u)$ over $(0, t)$ is at $u = t - t_c$, so the optimal choice of slot pressure function is $f(t) = \delta(t - t_c)$. This is a result of the inertia effect which causes the mass transfer to ‘overshoot’, so more mass flow may be sucked into the slot over a shorter time than over the longer time, after which injection takes place.

Other variational problems may be posed by considering a ‘cost’ proportional to the integral of the square or some other function of f.

4.2. Results for other slot pressure functions

In many practical circumstances, it is not feasible to exert detailed control over the slot pressure, as this may require complicated pressure variation mechanisms. For systems that possess only two suction-pressure settings, the relevant slot pressure function is given by $f(t) = H(t)$. In other common cases where the slot pressure may be similarly ‘switched on’ but then powers up gradually, it is more realistic to consider the slot pressure function $f(t) = tH(t)$. We may analyse both cases using the transform solution described above.

For the first of these cases, we note that since the Laplace transform of $H(t)$ is $1/p$, the Laplace transform of $M_0(t)$ has a simple pole at the origin as well as at the simple poles $p_i = -\lambda_i \pm i\mu_i$. From (10) the residue of $e^{pt}\overline{M}_0(p)$ at $p = 0$ is equal to $\frac{1}{4}\pi$. Hence

$$M_0(t) = \tfrac{1}{4}\pi H(t)\left(1 + \sum_{i=1}^{\infty} e^{-\lambda_i t}(\alpha_i \cos\mu_i t + \beta_i \sin\mu_i t)\right), \tag{20}$$

where the λ_i and μ_i are as before, but the α_i and β_i are given by

$$\alpha_i - i\beta_i = 4\frac{1}{p_i}\frac{I_0(\frac{1}{2}p_i) + I_1(\frac{1}{2}p_i)}{I_0(\frac{1}{2}p_i)\log 2(2 + p_i) + I_1(\frac{1}{2}p_i)(1 + p_i \log 2)}.$$

Figure 4 shows the behaviour of the mass-flow function for the case $f(t) = H(t)$. In particular, since the function shown is the integral of the solution for $f(t) = \delta(t)$, it follows that the solution is non-monotonic, and reaches a maximum which is greater than the asymptotic value of $\frac{1}{4}\pi$ at $t = t_c$. This has further implications for a slot pressure function $f(t) = 1 - H(t)$. This corresponds to a case where the slot pressure is maintained below the external pressure and then suddenly becomes equal to it. In this case the fact that there is a maximum for the solution for $H(t)$ implies that the mass-transfer function will reach zero in a finite time, and then, according to the equations given, become negative. As soon as the mass transfer becomes negative, the model is invalid, as there is now a thin layer of fluid downstream that influences the flow. This result shows that injection may occur, even if $f(t)$ remains negative and finite for all t.

For the case $f(t) = tH(t)$, the mass flow is given by

$$M_0(t) = \tfrac{1}{4}\pi H(t)\left(t + \tfrac{1}{4} - \log 2 + \sum_{i=1}^{\infty} \Re\left[e^{(-\lambda_i + i\mu_i)t}\frac{4i}{p_i^2(ip_i\log 2 + e^{-p_i}(\log 2 - \frac{1}{2}))}\right]\right), \tag{21}$$

thus confirming that for large t, the solution resembles that calculated earlier for $f(t) = t$. Of course, for very large values of t the initial assumption that $f(t)$ was $O(1)$ is violated.

Finally, it is worth pointing out that the 'rim seal' case studied earlier may also be analysed when the turbine blades are impulsively started, so that $f(t) = \sin\nu t H(t)$. The solution is now

$$M_0(t) = A_\nu \cos\nu t + B_\nu \sin\nu t + \tfrac{1}{4}\pi\sum_{i=1}^{\infty}\Re\left[e^{(-\lambda_i + i\mu_i)t}\frac{4i\nu}{(p_i^2 + \nu^2)(ip_i\log 2 + e^{-p_i}(\log 2 - \frac{1}{2}))}\right],$$

where the A_ν and B_ν are given by (16) and (17), respectively.

In both cases the solutions decay quickly to the steady state solution as t tends to infinity (see Figure 5 which compares the mass flow for $f(t) = 1 + H(t)\sin t$ with that for $f(t) = 1 + \sin t$). For large t the difference between the unsteady solution and the steady solution is $O(e^{-\lambda_1 t})$, and so λ_1, which is given in Table 1, may be regarded as a 'universal decay constant'.

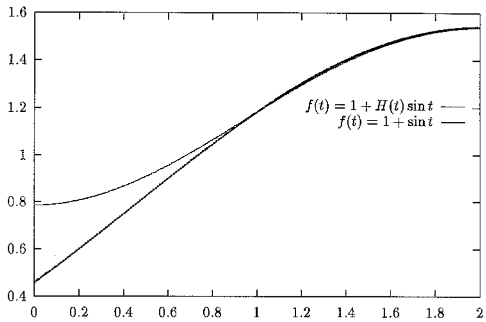

Figure 5. Mass transfer when $f(t) = 1 + H(t)\sin t$ and $f(t) = 1 + \sin t$.

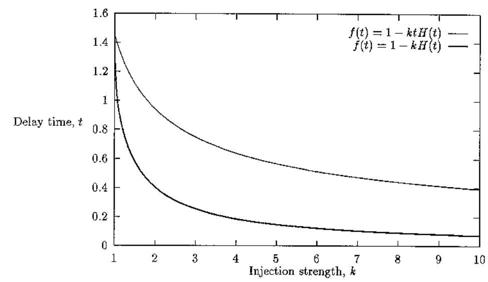

Figure 6. The time delay before injection as a function of injection strength.

Solutions where the slot pressure exceeds the external pressure are also of interest, for example in the 'rim seal' problem. When this happens, there is a time delay between the instant when the slot pressure exceeds the external pressure and the commencement of injection. This time delay is plotted against injection strength k for slot pressure profiles $f(t)$ proportional to $1-kH(t)$ in Figure 6. For comparison, the results plotted for pressure profiles proportional to $1-ktH(t)$ are also plotted. In an industrial application, the delay time is a crucial parameter, since it specifies how long an 'overpressure' may be tolerated before suction is halted and injection begins.

5. Evaluation of the height of the shear layer

Further physical insight into the flows that result from unsteady suction may be gained by evaluation of the height $S(x,t)$ of the shear layer that separates the ingested flow from the stagnant region in the slot. Given the mass flow, we may evaluate $S(x,t)$ from (8), using the fact that $S(x,t) = \sigma_x(x,t)$. For example, in the 'switched on suction' case when $f(t) = H(t)$, the expression for the mass flow $M_0(t)$ is given in Section 3.1, and hence

$$\begin{aligned} S(x,t) = &-\tfrac{1}{2}H(t)(a'(x) - H(x-t)(a'(x-t) + ta''(x-t)) \\ &- \tfrac{1}{4}tH(t)H(x-t)b'(x-t) \\ &- \frac{1}{4}\sum_{i=1}^{\infty}\left[\int_0^x \int_0^{x_1} (-\lambda_i^2 - \mu_i^2)\,e^{-\lambda_i(x_2-x+t)}\,\cos\mu_i(x_2-x+t)b(x_2)\,dx_2\,dx_1\right. \\ &\qquad - \int_0^x (\lambda_i \cos\mu_i(x_1-x+t) + \mu_i \\ &\qquad \left.\times \sin\mu_i(x_1-x+t))\,e^{-\lambda_i(x_1-x+t)}b(x_1)\,dx_1\right], \end{aligned}$$

where

$$a(x) = \tfrac{1}{2}\sqrt{x}\sqrt{1-x}(x-\tfrac{3}{2}) + \arcsin\sqrt{x}(\tfrac{3}{4} - x).$$

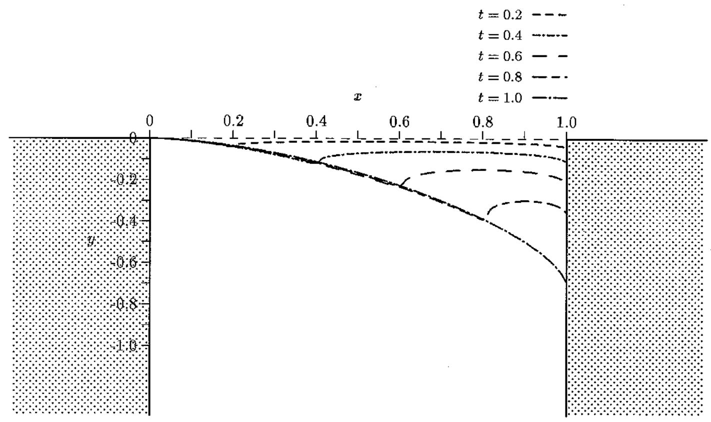

Figure 7. $S(x, t)$ for slot suction with $f(t) = H(t)$.

We may simplify this slightly by reversing the order of integration in the double integral and observing that the integrand does not depend upon x_1. Thus

$$S(x,t) = -\tfrac{1}{2}H(t)(a'(x) - H(x-t)(a'(x-t) + ta''(x-t))$$
$$- \tfrac{1}{4}tH(t)H(x-t)b'(x-t) + \frac{1}{4}\sum_{i=1}^{\infty}\int_0^x \mathrm{e}^{-\lambda_i(x_1-x+t)}b(x_1)$$
$$\times \{((1-x_1)(\lambda_i^2+\mu_i^2)+\lambda_i)\cos\mu_i(x_1-x+t) + \mu_i \sin\mu_i(x_1-x+t)\}\,\mathrm{d}x_1.$$

This function is plotted in Figure 7 for $t = 0{\cdot}2, (0{\cdot}2), 1$. When $t = 1$, the curve is almost indistinguishable from the steady state streamline given in Dewynne *et al.* [2].

In interpreting Figure 7, we should remember that, since the flow is unsteady, $S(x, t)$ is not a flow streamline. The slope discontinuities in S therefore do not imply the existence of discontinuous fluid velocities. It is also worth noting that, for the purposes of calculating $S(1, t)$ (the point of attachment to the downstream slot wall), the integrals in the solution for $S(x, t)$ may be expressed as products of exponential and Bessel functions. However, the resulting expressions are unwieldly and so are not included.

6. Asymptotic behaviour as $t \to 0$

For a system that is steady for $t < 0$ the behaviour for small t is of interest. We may derive this by using (3) and (7). For example, if $f(t) \sim t^n$ as $t \to 0$ for some constant $n > 0$, then if $\sigma \sim t^m$, say, (3) implies that $M_0 \sim t^{m-1}$, and so from (7) it follows that $m - 2 = n$. Thus, for cases where the difference between the slot pressure and the external pressure behaves asymptotically for small t as t^n, $S \sim t^{n+2}$ and $M_0 \sim t^{n+1}$.

A dimensional argument leads to the same result: the dimensions of M_0 are $LU_\infty = L^2/T$ (with T denoting time), the dimension of the height of the shear layer is L, and the dimension of the pressure difference is $\rho U_\infty^2 = M^2/T^2$ (with M denoting mass). Hence, it is to be expected that changes in the mass flow are slower than changes in the pressure difference and faster than the changes in $S(x, t)$ by a factor of T. Note that all changes still occur over a time scale of L/U_∞, as this is the natural time scale of the flow. Nevertheless, there is a significant time delay evident from the fact that for small t, the mass flow is a factor of t smaller than the slot pressure.

As far as the delay time (as described above) is concerned, these results imply that for large k the delay time tends to zero. Thus, $M_0'(t)$ is approximately constant up to the delay time. Since the solution for $f(t) = H(t)$ is the integral of the solution for $f(t) = \delta(t)$, namely $\zeta(t)$, it follows that for small t, the solution for $f(t) = 1 - kH(t)$ is $\frac{1}{4}\pi - k\zeta(0)t$, so in the limit as k tends to infinity the delay time tends to $\pi/(4\zeta(0)k)$.

7. Conclusions

For inviscid unsteady slot suction with constant total pressure at infinity and a given slot pressure that is independent of x, closed form expressions have been found for the mass flow. From these it is straightforward to obtain expressions for the other variables in the problem, such as the wall pressure and velocity perturbation. In particular, it may be shown for how long the slot pressure must exceed the external pressure before the slot begins to inject fluid into the free stream. It has also been shown that, because of the inertia of the fluid, it is possible for a slot pressure that is less than the external pressure to inject a small amount of fluid into the slot over a short period of time.

It is worth giving some attention to the exact nature of the conditions far upstream of the slot that have been used in the model discussed above. With a given slot pressure, mass flows have been calculated on the assumption that the total pressure and velocity are constant at infinity. An inevitable result of this is that the difference between the static pressure at a point $(1 - X, 0)$ far from the slot and the static pressure in the slot is $O(\log X)$ whenever the mass flow into the slot is not constant. This is because a time-dependent source cannot exist in a two-dimensional flow without an infinite pressure difference. Another interpretation, which entails some minor changes to the modelling, would be required to predict nonzero mass flows when both the static pressure in the slot and the static pressure at infinity are specified. One possibility in this case is to assume that the static pressure p_∞ and a free stream velocity U_∞ are prescribed at some *finite* upstream position $(1 - X, 0)$. The analysis is broadly the same as that undertaken in Section 2, provided that terms of order X^{-1} are ignored. Now there is no requirement for the pressure to become infinite for a time-dependent mass flow provided the $M_0'(t)\log(1-x^*)$ term in (5) remains finite, in which case it follows that $M_0'(t)$ is $O(1/\log X)$.

Other embellishments may also be made to the basic model. For example, the slot pressure could be specified at the 'bottom' ($y \to -\infty$) of the slot. In this case it might be expected that the pressure at the top of the slot will be a function of x^* as well as t^*. In order to evaluate this expression it will be necessary to solve Laplace's equation in the slot region, with zero mass flow through the boundaries $x^* = 0$ and $x^* = 1$, zero velocity at the bottom of the slot and the velocity at the top of the slot given by a matching condition. In this case continuity of pressure across the shear layer will give a singular integro differential equation similar to (6), though the details are involved. Angled slots, slightly compressible free-stream fluid and other complications may also be dealt with fairly easily.

As far as the rim-seal problem is concerned, the major item of interest to gas turbine designers is the amount of fluid (which in reality is hot, damaging gas) that is ingested. At present the consequences of nonnegligible amounts of suction are considered so undesirable that most turbine designs include provision for gas injection from rim seals. This gives rise to a more challenging problem than the one considered here where both suction and injection phases are present. Although attempts are currently being made to analyse this scenario, the details of the change from injection to suction are very complicated.

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